SOME MODERN METHODS AND PRINCIPLES OF GEOMETRY.*

By PROFESSOR HEINRICH MASCHKE, Ph. D.

It might be said of the most important parts of recent geometry that one conception dominates everywhere: that is the conception of the *group*. Suppose we are given a set of operations of any kind, which I call S_1, S_2, S_3, S_4, ... , finite or infinite in number, a set of operations which are defined by some law. Take now one of the operations, say S_i, apply it first, and after that has been done apply in succession another of the operations, S_k. If now it is so that the combined operation S_iS_k, which is obtained by applying first S_i and afterwards S_k, is again an operation *in the original set;* and if this is so for any two operations of the set, then the set forms a *group*. Let me give you an example. Think of a sphere with center fixed, and define a set of operations by all the possible rotations of the sphere about its center. That is an infinite number of operations. These operations, I say, form a group. Revolve the sphere first about a certain diameter through a certain angle. This is one of the operations of our set. After that has been done, take another axis and revolve the sphere about this second axis through a certain angle. Then it can be proved that the combined effect of these two rotations is equivalent to a single rotation about a certain axis and through a certain angle. The effect produced by two operations of the set applied in succession is the same as the effect of another operation contained in the set. Therefore, all these rotations form a group. The number of operations in this group is infinite.

Suppose now we have a triangle with sides of two, three, and four feet in length. Whether we make an investigation about this triangle here in this room in Ryerson Laboratory, or over in Cobb Hall, say, the result is the same. This means that in geometry our investigations are independent of the location of our figures in space. In other words, if I make a certain investigation of a certain triangle and then move that triangle to some other place in space, I do not change anything of the character of the theorem. Now instead of saying that we will move our figure from one place to another, I will rather say that we move the whole of space by that same amount which will bring this figure into coincidence with the other figure; and so then the following statement will be clear: our geometrical theorems are not changed when we submit the whole of space to a certain motion. The truth of our geometrical theorems is independent of the motion of space. If we consider all the possible motions of the whole of space, then these motions form a group, because the application of two motions in succession is equivalent to one single motion. Every motion can be considered as a transformation in the following sense: Suppose we take a point, and fix it by some means, say by its coördinates x, y, z; then by any motion of the space the point

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(x, y, z) goes into another point (say x', y', z'); and so every point of space is *transformed* into some other point. What we consider is this transformation, this connection between the points in the old position and the new position. Now, whenever the notion of a group comes in there is always the question of what remains invariant under such a group. If we subject the space to all possible motions, the most important invariant is the distance between two points. Take any two points, A and B; however you may move your space by translation, or rotation, or whatever you like, the distance between A and B remains always the same: it is an *invariant*. Also the angle between any two lines is invariant under this group of all possible motions in space. Of course these are not the only invariants. Indeed, every geometrical property—the theorem that the three perpendiculars at the middle points of the three sides of a triangle meet in a point, and all similar theorems—is independent of the accidental location of the triangle in space; all these theorems have an invariant character.

Let us go a step further. Take some scalene triangle, ABC, and consider the symmetrical triangle $A'B'C'$—all sides and angles equal respectively, but lying in the opposite direction. It is possible to make them lie one on the other by a certain motion. Take the line of symmetry, and revolve the plane of the first triangle about this line; then this triangle will cover the other one. But such a motion is not possible if you allow only motion in the plane. Let us say the triangle $A'B'C'$ is obtained from ABC by a *reflection* on their line of symmetry. In space, take a certain plane and reflect our figures on this plane. An irregular tetrahedron goes by such a reflection into another precisely equal to the first; but it is not possible by any *motion* in space to bring the two tetrahedrons into coincidence with each other. It is like the difference between the right and left hands. It would be possible to bring them together by mere motion if we could go into a space of four dimensions,* but it is not possible in space of three dimensions; just as in the case of the two triangles, where it is not possible to bring them into coincidence by motion in a plane, but only by motion in space of three dimensions.

But now I say in our geometrical investigations it does not make any difference whether we consider a certain figure or a figure which is deduced from the first one by such a reflection.

Let us consider all possible reflections in space on all possible planes. The question is, do they form a group? The answer is, no, because one reflection on one plane changes a given tetrahedron into a symmetrical tetrahedron, and any other reflection on a second plane changes the second tetrahedron into its symmetrical tetrahedron, which is equal and equally directed to the first, so that by two successive reflections we do not get again a reflection, but something which is equivalent to a motion. If, however, we join to all possible motions of space all possible reflections, this totality again forms a group, because no matter how you combine any motions and reflections, you always get either a motion or a reflection: that is to say, you get again an operation of the set. What is invar-

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*As to the space of four dimensions, see the explanations given at the end of the paper.

iant under this group? The distance between any two points, the angle between any two lines, and in the third place, every elementary geomotrical theorem.

Again let us go a step further. Suppose we investigate a triangle with sides respectively two, three, and four feet in length. A teacher in Paris does not say *feet*, but twenty, thirty, forty centimeters—a different size; but the theorems which he deduces from his triangle are the same as the theorems which we deduce. In other words, for our elementary geometrical theorems the size is immaterial. We allow then an expansion or reduction in size, everything remaining similar, of course. To fix the ideas let us define such an expansion or reduction in this way: Take a fixed point, and join it to all points in space by lines called radii vectores, and change every radius vector, without changing the angles, in the ratio $l:n$; the effect will be the expansion or reduction of the whole of space in size. Now let us join to all operations of our group containing all possible motions and reflections all these expansions and reductions; the combined operations again form a group, and this group has been called by Klein the *principal group* of geometry. Our geometrical theorems then remain true under this principal group: that is to say, they remain true if we apply any one of the operations of this principal group—any motion and reflection, or any expansion or reduction in size.

If we ask about invariants, we see at once that under this group distance is not invariant. But the ratio of two distances is invariant; it remains, of course, invariant for every motion and every reflection, and also for every expansion or reduction. The angle between two lines is also an invariant under the principal group. With this conception of the principal group we might give the following definition of the subject-matter of elementary geometry. We might say it is the establishment and deduction of geometrical properties which remain unchanged under this principal group.

Let us now extend this group by joining other operations. We then come right into the midst of modern geometry. Take any plane figure in space, on the board, for instance, and now take a point not in the plane of the board, and join this point to all the points of your figure: let the point be your eye, say, and let the straight lines be the lines on which you look upon the different points. If now you take a plane and place that plane in any position between the point and the board, there results what is called a projection of the figure on the board on this new plane. Let A be a point in the plane of the board, and O your center of projection, and let the corresponding point in the second plane be A', the point of intersection of the plane with OA. Thus every point A goes into a definite point A'. How does this figure in the second plane differ from the figure in the first plane? Is the distance between two points preserved? Certainly not. Is the ratio of the distances of two points preserved? Certainly not in general. If you have the points A and B, and C in the middle, and project from the point O, the point C' will *not* be in the middle of $A'B'$, unless the two planes are parallel. The angles between any two lines are also changed. But there is another thing which remains invariant—the ratio of two ratios. Take the line AB and divide it at C and D. Then

A C D B $\dfrac{CA}{CB}:\dfrac{DA}{DB}$

is invariant under this projection. This is called the *cross-ratio* or *anharmonic-ratio* between these points. This projection, however, might be considered as a transformation of the plane. Take the second plane and place it on the first plane; then you have on the first plane a certain point A and its corresponding point A', B and its corresponding point B'; whence you have a transformation of the different points on that plane.

A similar transformation is possible in space; only to make that projection we have to take a point outside of space; that is, a point in the fourth dimension somewhere. From that point we project every point of our space into another three-dimensional space, and then bring that second space into coincidence with the first. Then you have the same relation as before—for every point A a new point A'.

Analytically this transformation is much simpler. It can be shown that the coördinates x', y', z' of the new points A' are rational linear functions of the coördinates x, y, z of the old points A. From these formulas it follows at once that all these transformations (they are called *projections* in the plane and *collineations* in space) form a *group.*

Apply to the x', etc., a collineation, and you get x'', etc., in terms of x, y, z, a formula of the same kind. And every formula of that kind gives a collineation. Therefore the totality of all collineations in space form a group. This group contains the principal group, because every motion, every reflection, and every expansion or reduction can always be expressed by a formula of the above kind. This is the *group of projective geometry.*

Here the distance is not any longer invariant, nor is the angle, nor is the ratio between two lines; but the cross-ratio is an invariant, indeed the most important one of this group of projective geometry. The subject matter of projective geometry is then the study of geometrical theorems which remain unchanged under this group.

There are many other possible transformations of space, and each is defined by a certain group. I mention the Cremona transformations, in which the coördinates of the new points are no longer linear, but higher rational functions of the old, and the old of the new. These transformations also form a group, and that group contains all the groups which we had before. Another very general transformation is the transformation which underlies the so-called *analysis-situs*—the investigation of all those geometrical properties which remain unchanged for every continuous *deformation.* By that I mean any deformation which is such that two points which are very near together remain very near together; such a transformation as is made by squeezing a rubber ball in your hand. This transformation is so general, one might think, that by this process we could change any figure into almost any other figure. But by squeezing a ring you can never make a sphere, and conversely, by that process of deforma-

tion you can never get a ring from a sphere. There are also several invariants under this transformation—the most important of which is the so-called *genus*.

There is another principle of modern geometry which I wish to point out in a few words. I have mentioned occasionally the *fourth dimension*. Now the new principle referred to is the free use of any number of dimensions in geometry. Since we are three dimensional beings, it is utterly impossible for us to see in our imagination any space of higher than three dimensions. The study of higher spaces is therefore, and can only be, purely analytical. We might also treat analytic geometry of three dimensions in a purely analytical way, leaving aside all geometrical notions. In this sense analytic geometry of three dimensions is simply the study of functions of three independent variables x, y, z. Likewise, analytic geometry of four dimensions is the study of functions of four independent variables x, y, z, w. But in this study we might borrow the phraseology from analytic geometry of three dimensions. We might talk of a plane, of a line, a point, a three-dimensional space in the space of four dimensions, meaning by these terms certain linear equations or systems of equations in x, y, z, w. One linear equation would represent a three-dimensional space; for instance, $w=0$ would represent the ordinary space of three dimensions. Two linear equations in x, y, z, w would represent a plane; etc. Reasoning by analogy from three-dimensional space will help us then considerably in our analytic study in four dimensions.

In a certain way, however, a direct geometrical insight into spaces of higher dimensions is possible. When we consider our ordinary space as consisting not—as we are accustomed to—of points as elements, but of straight lines, then it becomes at once a space of four dimensions, because a straight line is determined by four independent coördinates. Again, taking other simple figurations as elements of space, for instance, the sphere, the circle, or the general surface of the second order, we might endow our ordinary space with any number of dimensions we please.

In geometry of three dimensions there are only five *regular* bodies: the tetrahedron, the hexahedron, the octahedron, the dodekahedron, and the ikosahedron. If we wish to represent these regular figures of space in the plane, we take a plane and a point outside, and project on the plane the regular hexahedron, for example. In general, several of the projected edges will meet. But that can be easily avoided in the following way: Place the body under consideration on the plane, and take as point of projection a point above the middle point of one of the faces and not far from it, in such a way that the upper face is so projected that it includes all the other faces. Then no two projected edges meet. For instance, Fig. 1 shows the projection of the regular hexahedron.

Let us do the same thing in a higher space. Take the space of four dimensions. It can be shown that in this space there are six regular bodies. Our imagination fails of course to see them, but we can see the projections of these bodies into our space of three dimensions. As the center of projection, we take

a point in the space of four dimensions chosen so that no meeting of the various lines occur.

A body of four dimensions is bounded by what corresponds to faces in the body of three dimensions —*i. e.*, by a certain number of bodies of three dimensions, in such a way that all these different bodies lie in different spaces; and every one of these is bounded by planes, every plane by edges, and every edge by vertices.

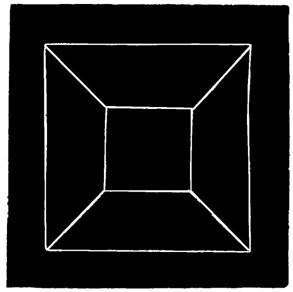

Fig. 1.

In Fig. 2 is given a perspective view of the projection of the so-called 8-cell, one of the regular bodies in four-dimensional space. We observe in the figure eight hexahedrons (counting also the one which includes all the others); these are the projections of the three-dimensional bodies (cells) which bound the four-dimensional body.

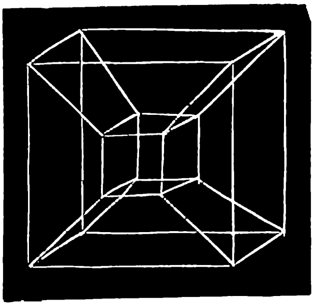

Fig. 2.

In the lecture itself, a set of wire models belonging to the mathematical department of the University of Chicago was shown to illustrate the projections into space of three dimensions of all six regular four-dimensional bodies.

The University of Chicago, October, 1902.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

148. Proposed by R. D. BOHANNAN, Ph. D., Professor of Mathematics, Ohio State University, Columbus, O.

If $\dfrac{x}{a+\alpha}+\dfrac{y}{b+\beta}+\dfrac{z}{c+\gamma}=1$, $\dfrac{x}{a+\beta}+\dfrac{y}{b+\beta}+\dfrac{z}{c+\beta}=1$, $\dfrac{x}{a+\gamma}+\dfrac{y}{b+\gamma}+\dfrac{z}{c+\gamma}=1$, show, without solving, that $x+y+z=a+\alpha+b+\beta+c+\gamma$.

Solution by JAMES McMAHON, A. M., Professor of Mathematics, Cornell University, Ithaca, N. Y.

There is probably a misprint in the first equation. It should be

$$\frac{x}{a+\alpha}+\frac{y}{b+\alpha}+\frac{z}{c+\alpha}=1.$$